REAL ANALYSIS
TOPIC XI - THE HEINE-BOREL THEOREM

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Abstract. Partially from

- https://math.stackexchange.com/questions/188996/showing-that-0-1-is-compact
- https://math.stackexchange.com/questions/567335/cartesian-product-of-compact-sets-is-compact

Definition 1. Let X be a topological space and let $K \subset X$.
A *cover* of K is a collection $\mathcal{C} \subset X$ such that $X \subset \cup\mathcal{C}$.
Let $\mathcal{C}$ be a cover of K.
We say that $\mathcal{C}$ is a *finite cover* if $\mathcal{C}$ contains finitely many sets.
We say that $\mathcal{C}$ is an *open cover* if every set in $\mathcal{C}$ is open.
We say that $\mathcal{D} \subset \mathcal{C}$ is a *subcover* of $\mathcal{C}$ if $X \subset \cap\mathcal{D}$.
We say that K is *compact* if every open cover of K has a finite subcover.

The concept of compactness has its origins in proofs around the end of the eighteenth century that continuous functions on closed intervals are uniformly continuous. It has since become an important topological invariant for classification of more abstract spaces. Invariance is a consequence of the next proposition, which says that the continuous image of a compact set is compact.

Proposition 1. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $K \subset X$ be compact. Then $f(K)$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(K)$. Define

$$\mathcal{U} = \{U \subset X \mid U = f^{-1}(V) \text{ for some } V \in \mathcal{V}\}.$$

Clearly $K \in \cup\mathcal{U}$, and since f is continuous, each set in $\mathcal{U}$ is open, so $\mathcal{U}$ is an open cover of K. Thus, $\mathcal{U}$ has a finite subcover, say

$$\{U_1, \dots, U_n\} \subset \mathcal{U}.$$

For each $i = 1, \dots, n$, there exists $V_i \in \mathcal{V}$ such that $U_i = f^{-1}(V_i)$. Since $K \subset \cup_{i=1}^n U_i$, we see that $f(K) \subset \cup_{i=1}^n V_i$. Thus, $\{V_1, \dots, V_n\}$ is an open cover of $f(K)$, which shows that $f(K)$ is compact. $\square$

The following result is interesting in its own right and is useful in what follows.

Proposition 2. *Let X be a topological space. Let $K \subset X$ be compact, and let $F \subset K$ be closed. Then F is compact.*

Proof. Let $\mathcal{C}$ be an open cover of F, and let $U = F^c$. Since F is closed, U is open. Moreover, $\mathcal{C} \cup \{U\}$ covers X, so it also covers K. As such, since K is compact, $\mathcal{C} \cup \{U\}$ has a finite subcover of K, say $\mathcal{D}$ is a finite open cover of K. Then, $\mathcal{D}$ also covers F. Note that U is unnecessary to help cover F, and indeed, $\mathcal{D}\{U\}$ is a finite subcover of $\mathcal{C}$. Thus F is compact. $\square$

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Compactness is an abstract concept which models some important aspects of certain subsets of a more concrete topological space, the metric space $\mathbb{R}^n$. The *Heine-Borel* theorem states that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. We prove this theorem in parts; the forward direction can be stated for an arbitrary metric space.

Definition 2. Let X be a metric space and let $D \subset X$. We say that D is *bounded* if there exists $a \in X$ and $r \in \mathbb{R}$ such that $D \subset B_r(a)$, where

$$B_r(a) = \{x \in X \mid d(a, x) < r\}.$$

Proposition 3. *Let (X, d) be a metric space, and let $K \subset X$. If K is compact, then K is bounded.*

Proof. If K is empty, it is bounded, so we assume that K is nonempty. Let $x \in K$. Let

$$\mathcal{C} = \{B_r(x) \mid r \in \mathbb{R}\}.$$

The sets in $\mathcal{C}$ are open, and $K \subset X = \cup\mathcal{C} = X$, so $\mathcal{C}$ is an open cover of K, so $\mathcal{C}$ has a finite subcover, say $\mathcal{D} \subset \mathcal{C}$. Let $R = \max\{r \in \mathbb{R} \mid B_r(x) \in \mathcal{D}\}$. Then $K \subset B_R(x)$, so K is bounded. □

Proposition 4. *Let X be a Hausdorff space, and let $K \subset X$ be compact. Then K is closed.*

Proof. We wish to show that $X \smallsetminus K$ is open. Let $x \in X \smallsetminus K$; it suffices to show that x has an open neighborhood which is disjoint from K.

For each $y \in K$, there exists disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. The collection $\mathcal{V} = \{V_y \mid y \in K\}$ is an open cover of K, and since K is compact, it has a finite subcover, say $\{V_{y_1}, \dots, V_{y_n}\}$. Then $\cap_{i=1}^n U_{y_i}$ is an open neighborhood of x which is disjoint from K.

We may demonstrate disjointness via DeMorgan's laws, as follows. For $A \subset X$, let $A^c = X \smallsetminus A$ denote the complement of A. Note that for $A \subset B$ we have $B^c \subset A^c$. We see that $U_y \subset V_y^c$. Now $K \subset \cup_{i=1}^n V_{y_i}$, so

$$K^c \supset \Big(\bigcup_{i=1}^n V_{y_i}\Big)^c = \bigcap_{i=1}^n V_{y_i}^c \supset \bigcap_{i=1}^n U_{y_i},$$

so $\cap_{i=1}^n U_{y_i}$ is disjoint from K. □

Example 1. We give an example of a topological space which contains a compact set which is not closed.

Let $I = [0, 1]$, and let $X = I \cup \{0'\}$, where $0'$ is an alternate version of zero. The topology of X is generated by the open sets in I, together with the open neighborhood of $0'$, which are sets of the form $\{0'\} \cup (0, a)$ for $a \in (0, 1)$. This is a T_1 space which is not Hausdorff. Then $I \subset X$ is compact, but not closed, since its complement $\{0'\}$ is not open.

Of course, this example requires knowledge that I is compact. The proof of this follows.

Proposition 5. *The closed unit interval $I = [0, 1]$ is compact.*

Proof. Let $\mathcal{O}$ be an (arbitrary!) open cover. Let P be the set of points x in $[0, 1]$ such that $[0, x]$ can be covered by finitely many elements of $\mathcal{O}$. Under the convention that $[0, 0] = \{0\}$, we have $0 \in P$ and P is bounded above by 1. Therefore, P has a supremum s.

We first show that $[0, s]$ can be covered by finitely many sets in $\mathcal{O}$. This is trivial when $s = 0$, so assume $s > 0$. Let $O_s \in \mathcal{O}$ be a set containing s. Then there is an $\epsilon \in (0, s)$ such that $(s - \epsilon, s] \subseteq O_s$. By assumption, there is a finite subcover of $[0, s - \epsilon/2]$. By adding O_s to that finite subcovering, we get a finite subcovering of $[0, s]$.

We now show that $s = 1$. Suppose $s < 1$ and let $O_s \in \mathcal{O}$ be a set containing s. Then there is an $\epsilon > 0$ such that $[s, s + \epsilon) \subseteq O_s$. So taking a finite subcover of $[0, s]$ and adding the set O_s gives us a finite subcover of $[0, s + \epsilon/2]$, contradicting the construction of s.

□

Next, we wish to show that the product of two compact spaces is compact. This is most easily expressed using the following lemma, which is a version of the famous "Tube Lemma".

Lemma 1. *Let X and Y be topological spaces, with Y compact. Let $U \subset X \times Y$ be an open set. Let*

$$V = \{x \in X \mid \{x\} \times Y \subset U\}.$$

Then V is open in X.

Proof. Sets of the form $D \times E$, where $D \subset X$ and $E \subset Y$, form a basis for the topology of $X \times Y$.

Let $x \in V$, so that $\{x\} \times Y \subset U$. For each $y \in Y$, there exist open sets $D_y \subset X$ and $E_y \subset Y$ such that $(x, y) \in D_y \times E_y \subset U$. Since Y is compact, there is a finite set $\{y_1, \ldots, y_k\} \subset Y$ such that $Y \subset \cup_{i=1}^k E_y$. Set $N_x = \cap_{i=1}^k D_{y_i}$, so that N_x is an open subset of X. Moreover,

$$N_x \times Y \subset \bigcup_{i=1}^{k} (N_x \times E_{y_i}) \subset \cup_{i=1}^k (D_{y_i} \times E_{y_i}) \subset U,$$

so $N_x \subset V$. Thus $V = \cup_{x \in V} N_x$ is open in X. □

Proposition 6. *Let X and Y be compact topological spaces. Then $X \times Y$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$; we wish to show that $\mathcal{U}$ has a finite subcover.

For each $x \in X$, let $Y_x = \{x\} \times Y$. Clearly Y_x is homeomorphic to Y, and so it is compact. The set in $\mathcal{U}$ is an open cover of Y_x, so it admits a finite subcover. Thus for each $x \in X$, let $\mathcal{U}_x \subset \mathcal{U}$ be finite and cover Y_x.

Now, for each $x \in X$, let

$$V_x = \{x' \in X \mid Y_{x'} \subset \cup \mathcal{U}_x\}.$$

By Lemma 1, V_x is open in X, and $x \in V_x$. Thus $\{V_x \mid x \in X\}$ is an open cover of X, and since X is compact, there exists a finite set $\{x_1, \dots, x_r\} \subset X$ such that $\{V_{x_1}, \dots, V_{x_r}\}$ covers X. Each $V_{x_i} \times Y$ is covered by finitely many sets from $\mathcal{U}$. Combining these collections gives a finite subset of $\mathcal{U}$ which covers $X \times Y$. □

Proposition 7. *The hypercube $[-a, a]^n \subset \mathbb{R}^n$ is compact.*

Proof. There exists a continuous function $I \to [-a, a]$, so $[-a, a]$ is compact. Since the product of two compact sets is compact, it is clear by induction that the product of finitely many compact sets is compact. Thus, $[-a, a]^n$ is compact. □

Theorem 1. (Heine-Borel Theorem) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Let $K \subset \mathbb{R}^n$.

($\Rightarrow$) Suppose that K is compact. Then, since $\mathbb{R}^n$ is a metric space, K is bounded. Also, since $\mathbb{R}^n$ is Hausdorff, K is closed.

($\Leftarrow$) Suppose that K is closed and bounded. Since K is bounded, there exists $a \in \mathbb{R}$ such that $K \subset [-a, a]^n$, which is compact. Thus, K is a closed subset of a compact set, and so K is also compact. □

Exercise 1. Let $\mathbb{R}^\infty$ denote the set of all sequences of real numbers which are eventually zero, that is, sequences $\vec{x} = (x_n)$ such that $x_n = 0$ for all but finitely many n. Let $X = \mathbb{R}^\infty$ and for $\vec{x}, \vec{y} \in X$, define

$$d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $\vec{x} = (x_n)$ and $\vec{y} = (y_n)$. This make sense without considering convergence, since there are only finitely many nonzero summands. Then (X, d) is a metric space. Let $|\vec{x}| = d(\vec{x}, \vec{0})$. Show that

$$D = \{\vec{x} \in \mathbb{R}^\infty \mid |\vec{x}| \leq 1\}$$

is closed and bounded but not compact.

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